

A Heterotic $N=2$ String with Space-Time Supersymmetry

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Abstract

We reconsider the issue of embedding space-time fermions into the four-dimensional $N=2$ world-sheet supersymmetric string. A new heterotic theory is constructed, taking the right-movers from the $N=4$ topological extension of the conventional $N=2$ string but a $c=0$ conformal field theory supporting target-space supersymmetry for the left-moving sector. The global bosonic symmetry of the full formalism proves to be $U(1,1)$, just as in the usual $N=2$ string. Quantization reveals a spectrum of only two physical states, one boson and one fermion, which fall in a multiplet of $(1,0)$ supersymmetry.

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1. Introduction

The $N=2$ string has attracted considerable interest over the past decade (see [1] for an older review and [2] for BRST quantization). Being relatively simple to analyze, the theory displays a few peculiar properties which distinguish it among others. To mention only the most significant points, there appears only a finite number of physical states in the excitation spectrum. However, one encounters a continuous family of sectors interpolating between R and NS and being connected by spectral flow [3]. The latter is essentially a consequence of the complex geometry intrinsic to the RNS formulation of the $N=2$ string (see e.g. [4]). A remnant of the gauged $U(1)$ R symmetry, spectral flow enables one to restrict oneself to a preferred sector, say the NS one. All tree-level amplitudes with more than three external legs prove to vanish [5, 6] (for details of loop calculations see a recent work [7] and references therein), in complete agreement with the fact that the massless excitations of the $N=2$ string parametrize self-dual gauge or gravitational theory in two spatial and two temporal dimensions [6]. Also worth mentioning is a “universal string” interpretation deriving from the fact that $N=0$ and $N=1$ strings can be viewed as special classes of vacua of the $N=2$ string [8].

On the downside, the Brink-Schwarz action [9] lacks manifest $SO(2,2)$ Lorentz covariance, and the world-sheet theory of $N=2$ supergravity coupled to matter supermultiplets is unable to produce space-time fermions in the excitation spectrum. An early attempt [10] to gain fermions by adding twisted sectors [11] to the $N=2$ string failed because the necessary GSO projection prevented any interaction between bosonic and fermionic states (see, however, [12] for a different strategy).

Alternatively, a manifestly space-time supersymmetric and $SO(2,2)$ covariant Green-Schwarz-type formulation of the $N=2$ string was proposed in [13]. Yet, as was later recognized [14, 15], the set of currents invented in [13] does not form a closed algebra. This drawback was overcome in [14, 15] by taking a smaller but closed subset of the constraints. However, the massless states of the resulting $N=2$ string, although consisting of a scalar and a spinor, do not interact according to self-dual super Yang-Mills or self-dual supergravity.

A more successful approach has been advocated by de Boer and Skenderis [16] who combined a right-moving heterotic $N=(1,2)$ string with a left-moving Green-Schwarz-Berkovits type sigma model [17]. Since, by the very construction, the $N=(1,2)$ string lives in a two- or three-dimensional target, the analysis of [16] is likely to yield only a dimensional reduction of self-dual supergravity to 2+1 dimensions. A common feature of both approaches is the introduction of momenta canonically conjugate to the space-time spinors and a manifest Lorentz covariance.

Quite recently, the zero-mode structure was investigated for a potential string theory which would be capable of describing supersymmetric self-dual Yang-Mills in 2+2 dimensions [18]. In agreement with [16] the analysis suggests that the ultimate formulation seems to be doubly supersymmetric, i.e. possessing both local world-sheet supersymmetry and kappa symmetry.

In the present paper we reconsider the issue of embedding space-time fermions into

the $N=2$ string. We take advantage of the existing literature [14, 16] and construct a new heterotic string in two spatial and two temporal dimensions. In our scenario self-duality is implemented by the right-movers while manifest target-space supersymmetry is captured by left-movers. The drawbacks of the previous attempts we discussed above suggest, however, that the presence of space-time fermions in the spectrum might be incompatible with manifest $SO(2,2)$ Lorentz invariance. Here, we choose to relax the latter property and build a new heterotic string which supports $U(1,1)$, just as the conventional $N=2$ string does.

Since there are two time-like directions in the target space one can introduce two light-cone structures. When specifying left-moving degrees of freedom we use the hitherto unexploited possibility to supersymmetrize only one light-cone direction just in the way it works in $(1,0)$ supergravity (see e.g. Ref. [19]). Remarkably, this can be done without spoiling the global $U(1,1)$ group which is installed in the formalism by the left-movers. The system of currents we use here looks very similar to that examined in [14] but differs in the global symmetry structure. As to the right-movers, a first guess would be to take those of the conventional $N=2$ string. Surprisingly, this proves to be incompatible with the $U(1,1)$ kept by the left-movers. In order to reconcile the two points we turn to the $N=4$ topological reformulation of the $N=2$ string proposed by Berkovits and Vafa [20] and further studied in [21], which thus specifies the right-moving sector of the model. Quantization of the theory reveals only two physical states in the spectrum, one boson and one fermion, which prove to fall in a multiplet of the $(1,0)$ space-time supersymmetry.

The organization of the paper is as follows. In Sect. 2 we briefly review the salient features concerning the conventional $N=2$ string and its $N=4$ topological reformulation. We also specify the right-movers and fix our notation. Sect. 3 contains a description of the left-moving sector and a realization of the space-time supersymmetry. The global symmetry structure of the complete model is discussed in detail in Sect. 4. In particular, a supersymmetric extension of the $U(1,1)$ group is given. Sects. 5 and 6 perform the quantization of the right- and left-movers, respectively. We summarize our results and discuss possible further developments in Sect. 7.

2. The right-moving sector

As has been discussed in the Introduction, the right-movers in the new model are designed to keep the self-duality for the massless states. Guided by the formalism for the conventional closed $N=2$ string, it seems natural to choose them to be just the right-movers of the latter. We proceed directly to the superconformal gauge. Then the space of histories is spanned by the canonical pair (z^a, p_{za}) , $a = 0, 1$, its complex conjugate $(\bar{z}^a, p_{\bar{z}a})$, and a couple of complex conjugate fermions $\psi^a, \bar{\psi}^a$. The canonical brackets imposed on the fields are

$$\begin{aligned} \{z^a(\sigma), p_z^b(\sigma')\} &= \eta^{ab}\delta(\sigma - \sigma'), & \{\bar{z}^a(\sigma), p_{\bar{z}}^b(\sigma')\} &= \eta^{ab}\delta(\sigma - \sigma'), \\ \{\psi^a(\sigma), \bar{\psi}^b(\sigma')\} &= i\eta^{ab}\delta(\sigma - \sigma'), \end{aligned} \tag{1}$$

with the Minkowski metric $\eta^{ab} = \text{diag}(-, +)$.

The dynamics in the sector is governed by the Hamiltonian

$$H = \int_0^{2\pi} d\sigma \left\{ 2\pi(p_z^a p_{\bar{z}a} + \frac{1}{(2\pi)^2} \partial_1 z^a \partial_1 \bar{z}_a) - \frac{i}{2} \bar{\psi}^a \partial_1 \psi_a - \frac{i}{2} \psi^a \partial_1 \bar{\psi}_a \right\}, \quad (2)$$

where ∂_1 denotes the derivative with respect to σ .

Given the space, one can build a set of currents (a contraction of indices is implied),

$$\begin{aligned} T &= (p_z + \frac{1}{2\pi} \partial_1 \bar{z})(p_{\bar{z}} + \frac{1}{2\pi} \partial_1 z) - \frac{i}{2\pi} (\psi \partial_1 \bar{\psi} + \bar{\psi} \partial_1 \psi) = 0, \\ G &= (p_z + \frac{1}{2\pi} \partial_1 \bar{z}) \psi = 0, \quad \bar{G} = (p_{\bar{z}} + \frac{1}{2\pi} \partial_1 z) \bar{\psi} = 0, \\ J &= \bar{\psi} \psi = 0, \end{aligned} \quad (3)$$

which is closed under the bracket (1) and forms nothing but the $N=2$ superconformal algebra.

Making use of the brackets (1), the equations of motion can be easily derived from the Hamiltonian (as usual $\partial_{\pm} = \partial_0 \pm \partial_1$),

$$\begin{aligned} \dot{z}^a - 2\pi p_{\bar{z}}^a &= 0, \quad \dot{p}_{\bar{z}}^a - \frac{1}{2\pi} \partial_1 \partial_1 z^a = 0, \\ \dot{\bar{z}}^a - 2\pi p_z^a &= 0, \quad \dot{p}_z^a - \frac{1}{2\pi} \partial_1 \partial_1 \bar{z}^a = 0, \\ \partial_- \psi^a &= 0, \quad \partial_- \bar{\psi}^a = 0. \end{aligned} \quad (4)$$

Being applied to the currents, these yield

$$\partial_- T = 0, \quad \partial_- G = 0, \quad \partial_- \bar{G} = 0, \quad \partial_- J = 0. \quad (5)$$

Notice further that the equations (4) are amended by the periodicity conditions

$$(\delta z \partial_1 \bar{z} + \delta \bar{z} \partial_1 z)|_0^{2\pi} = 0, \quad (\bar{\psi} \delta \psi + \psi \delta \bar{\psi})|_0^{2\pi} = 0. \quad (6)$$

The bosons describe string coordinates and are single-valued (periodic) in the flat space. Due to their complex nature, the NSR fermions live in a twisted spinor bundle [22], i.e.

$$\psi(\sigma + 2\pi) = e^{2i\nu\pi} \psi(\sigma), \quad \bar{\psi}(\sigma + 2\pi) = e^{-2i\nu\pi} \bar{\psi}(\sigma), \quad (7)$$

with ν being an arbitrary real number. For the currents this amounts to

$$\begin{aligned} G(\sigma + 2\pi) &= e^{2i\nu\pi} G(\sigma), \quad \bar{G}(\sigma + 2\pi) = e^{-2i\nu\pi} \bar{G}(\sigma), \\ T(\sigma + 2\pi) &= T(\sigma), \quad J(\sigma + 2\pi) = J(\sigma). \end{aligned} \quad (8)$$

The choice of the parameter ν determines the moding for the fermionic fields. This is in contrast with the conventional $N=1$ string for which only two options (R or NS) are available.

Varying ν changes the representation of the $N=2$ superconformal algebra, an effect known as “spectral flow”. However, all such representations are in fact equivalent [3]

Actually, it is straightforward to check that the continuous automorphism (preserving Eq. (5))

$$G' = e^{-i\alpha(\tau+\sigma)}G, \quad \bar{G}' = e^{i\alpha(\tau+\sigma)}\bar{G}, \quad T' = T - \frac{\alpha}{\pi}J, \quad J' = J, \quad (9)$$

does the job for an arbitrary real number α , since

$$G'(\sigma + 2\pi) = e^{2i(\nu-\alpha)\pi}G'(\sigma). \quad (10)$$

Because α is at our disposal, one can always stick with a preferred representation.⁴ For the rest of this paper we choose to work in the NS picture, thus just putting $\nu = \frac{1}{2}$ in Eq. (7) above.

It is worth noting further that, due to the complex structure intrinsic to the $N=2$ string, the (spin cover of the) full target-space Lorentz group $Spin(2,2) = SU(1,1) \times SU(1,1)'$ gets broken to $U(1) \times SU(1,1) \simeq U(1,1)$ which is then the global symmetry group of the formalism under consideration.

So far, our discussion made use of complex field variables. In the next section, where we shall introduce the left-movers, it will turn out that the formulation is more transparent in a specific real field basis. We thus devote the remnant of this section to conform the present analysis to a real notation.

Given a fermionic field ψ^a , with $a=0,1$, we first transform the vector index a to a “light-cone” basis,

$$\psi^\pm = \frac{1}{\sqrt{2}}(\psi^0 \pm \psi^1), \quad \bar{\psi}^\pm = \frac{1}{\sqrt{2}}(\bar{\psi}^0 \pm \bar{\psi}^1), \quad (11)$$

and then take the real and imaginary parts,

$$\psi^\pm = \varphi^\pm + i\chi^\pm, \quad \bar{\psi}^\pm = \varphi^\pm - i\chi^\pm, \quad (12)$$

to be the new field variables. The only nonvanishing brackets are

$$\{\varphi^+(\sigma), \varphi^-(\sigma')\} = -\frac{i}{2}\delta(\sigma - \sigma'), \quad \{\chi^+(\sigma), \chi^-(\sigma')\} = -\frac{i}{2}\delta(\sigma - \sigma'). \quad (13)$$

Analogously, for the bosonic variables one performs a similar coordinate change,

$$z^\pm = \frac{1}{\sqrt{2}}(z^0 \pm z^1), \quad \bar{z}^\pm = \frac{1}{\sqrt{2}}(\bar{z}^0 \pm \bar{z}^1), \quad (14)$$

and defines the new fields

$$z^\pm = X^\pm + iY^\pm, \quad p_z^\pm = p^\pm + ip_Y^\pm, \quad \bar{z}^\pm = X^\pm - iY^\pm, \quad p_{\bar{z}}^\pm = p^\pm - ip_Y^\pm. \quad (15)$$

The latter prove to obey

$$\begin{aligned} \{X^+(\sigma), p^-(\sigma')\} &= -\frac{1}{2}\delta(\sigma - \sigma'), & \{X^-(\sigma), p^+(\sigma')\} &= -\frac{1}{2}\delta(\sigma - \sigma'), \\ \{Y^+(\sigma), p_Y^-(\sigma')\} &= \frac{1}{2}\delta(\sigma - \sigma'), & \{Y^-(\sigma), p_Y^+(\sigma')\} &= \frac{1}{2}\delta(\sigma - \sigma'). \end{aligned} \quad (16)$$

⁴Because α is a $U(1)$ modulus, it must be considered beyond tree level [7].

It is now trivial to conform the Hamiltonian to the new notation,

$$H = \int_0^{2\pi} d\sigma \{ -4\pi p^+ p^- - 4\pi p_Y^+ p_Y^- - \frac{1}{\pi} \partial_1 X^+ \partial_1 X^- - \frac{1}{\pi} \partial_1 Y^+ \partial_1 Y^- + i\varphi^+ \partial_1 \varphi^- + i\varphi^- \partial_1 \varphi^+ + i\chi^+ \partial_1 \chi^- + i\chi^- \partial_1 \chi^+ \}. \quad (17)$$

Analogous manipulations with the set of currents, after some simple algebra, yield

$$\begin{aligned} \tilde{T} &= -2(p^- + \frac{1}{2\pi} \partial_1 X^-)(p^+ + \frac{1}{2\pi} \partial_1 X^+) - 2(p_Y^- - \frac{1}{2\pi} \partial_1 Y^-)(p_Y^+ - \frac{1}{2\pi} \partial_1 Y^+) + \\ &\quad \frac{i}{\pi} \varphi^+ \partial_1 \varphi^- + \frac{i}{\pi} \varphi^- \partial_1 \varphi^+ + \frac{i}{\pi} \chi^+ \partial_1 \chi^- + \frac{i}{\pi} \chi^- \partial_1 \chi^+ = 0, \\ \tilde{G} &= (p^+ + \frac{1}{2\pi} \partial_1 X^+) \varphi^- + (p^- + \frac{1}{2\pi} \partial_1 X^-) \varphi^+ - (p_Y^+ - \frac{1}{2\pi} \partial_1 Y^+) \chi^- - \\ &\quad (p_Y^- - \frac{1}{2\pi} \partial_1 Y^-) \chi^+ = 0, \\ \tilde{H} &= (p^+ + \frac{1}{2\pi} \partial_1 X^+) \chi^- + (p^- + \frac{1}{2\pi} \partial_1 X^-) \chi^+ + (p_Y^+ - \frac{1}{2\pi} \partial_1 Y^+) \varphi^- + \\ &\quad (p_Y^- - \frac{1}{2\pi} \partial_1 Y^-) \varphi^+ = 0, \\ \tilde{J} &= i\chi^+ \varphi^- + i\chi^- \varphi^+ = 0. \end{aligned} \quad (18)$$

We do not see the vector indices any more, and the invariance under $U(1,1)$ is kept by contracting a “+” with a “−” (for more details see Sect. 4). The only nonvanishing brackets in practical calculations are those involving plus and minus indices simultaneously. In view of implementing global supersymmetry in the target, one can now see an intriguing possibility: One may try to supersymmetrize in only *half* the spatial directions, making recourse to $(1,0)$ superspace (see e.g. Ref. [19])! Thus, our main reason to prefer a real notation is the greater transparency of the target-space supersymmetry in this basis.

No work has to be done with the equations of motion, these just acquiring the form

$$\begin{aligned} \dot{X}^\pm - 2\pi p^\pm &= 0, \quad \dot{p}^\pm - \frac{1}{2\pi} \partial_1 \partial_1 X^\pm = 0, \\ \dot{Y}^\pm + 2\pi p_Y^\pm &= 0, \quad \dot{p}_Y^\pm + \frac{1}{2\pi} \partial_1 \partial_1 Y^\pm = 0, \\ \partial_- \varphi^\pm &= 0, \quad \partial_- \chi^\pm = 0. \end{aligned} \quad (19)$$

Recalling that we can always stick with the NS periodicity conditions thanks to spectral flow, we finally write down the general solution in the right-moving sector

$$\begin{aligned} X^\pm(\tau, \sigma) &= x^\pm + \tau p^\pm + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} a_n^\pm e^{-in(\tau-\sigma)} + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} b_n^\pm e^{-in(\tau+\sigma)}, \\ Y^\pm(\tau, \sigma) &= y^\pm + \tau p_Y^\pm + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \tilde{a}_n^\pm e^{-in(\tau-\sigma)} + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \tilde{b}_n^\pm e^{-in(\tau+\sigma)}, \\ \varphi^\pm &= \frac{1}{2\sqrt{\pi}} \sum_{r \in \mathbb{Z}+1/2} f_r^\pm e^{-ir(\tau+\sigma)}, \quad \chi^\pm = \frac{1}{2\sqrt{\pi}} \sum_{r \in \mathbb{Z}+1/2} g_r^\pm e^{-ir(\tau+\sigma)}. \end{aligned} \quad (20)$$

The index “+” is not to be confused with the hermitian conjugation which in the following we denote by “*”.

The reality condition for the fields together with the brackets (13) and (16) specify the hermiticity properties and the Poisson brackets for the Fourier modes in the usual way,

$$\begin{aligned} \{a_n^-, a_m^+\} &= in\delta_{n+m,0}, & \{a_n^+, a_m^-\} &= in\delta_{n+m,0}, & (a_n^\pm)^* &= a_{-n}^\pm, \\ \{x^-, p^+\} &= -\frac{1}{2}, & \{x^+, p^-\} &= -\frac{1}{2}, \\ \{f_r^+, f_q^-\} &= -i\delta_{r+q,0}, & (f_r^\pm)^* &= f_{-r}^\pm. \end{aligned} \quad (21)$$

The modes $b_n^\pm, \tilde{a}_n^\pm, \tilde{b}_n^\pm, g_r^\pm, y^\pm, p_y^\pm$ satisfy precisely the same relations.

It is noteworthy that the $N=2$ superconformal currents given in Eq. (3) above are not the maximal closed set one can realize on the matter fields. As was pointed out by Siegel [23] and later on by Berkovits and Vafa [20], two more bosonic currents (of spin 1 before twisting) and two more fermionic ones (of spin 3/2),

$$\epsilon^{ab}\psi_a\psi_b = 0, \quad \epsilon^{ab}\bar{\psi}_a\bar{\psi}_b = 0, \quad \epsilon^{ab}(p_{\bar{z}a} + \frac{1}{2\pi}\partial_1 z_a)\psi_b = 0, \quad \epsilon^{ab}(p_{za} + \frac{1}{2\pi}\partial_1 \bar{z}_a)\bar{\psi}_b = 0, \quad (22)$$

or, in real notation,

$$\begin{aligned} \tilde{G}_1 &= (p^- + \frac{1}{2\pi}\partial_1 X^-)\chi^+ - (p^+ + \frac{1}{2\pi}\partial_1 X^+)\chi^- - (p_Y^- - \frac{1}{2\pi}\partial_1 Y^-)\varphi^+ + \\ &\quad (p_Y^+ - \frac{1}{2\pi}\partial_1 Y^+)\varphi^- \\ \tilde{H}_1 &= -(p^- + \frac{1}{2\pi}\partial_1 X^-)\varphi^+ + (p^+ + \frac{1}{2\pi}\partial_1 X^+)\varphi^- - (p_Y^- - \frac{1}{2\pi}\partial_1 Y^-)\chi^+ + \\ &\quad (p_Y^+ - \frac{1}{2\pi}\partial_1 Y^+)\chi^-, \\ \tilde{J}_1 &= i\chi^-\varphi^+ - i\chi^+\varphi^- = 0, \quad \tilde{J}_2 = i\varphi^+\varphi^- - i\chi^+\chi^- = 0, \end{aligned} \quad (23)$$

extend the algebra to a “small” $N=4$ superconformal algebra. We remark that the triplet of spin-one currents $(\tilde{J}, \tilde{J}_1, \tilde{J}_2)$ forms an $su(1,1)$ subalgebra. Furthermore, adding the new currents enlarges the symmetry of the formalism. Besides the $U(1,1)$ group which comes with the $N=2$ string currents and acts as a trivial automorphism on the $N=4$ superconformal algebra just leaving each current invariant, there appears an extra external automorphism group $U(1,1)$ which maps $\tilde{G}, \tilde{H}, \tilde{G}_1, \tilde{H}_1$ generators one into another. Before twisting, the symmetry of the topological extension is $U(1,1) \times U(1,1)$ which includes the full Lorentz group $SO(2,2)$. After a topological twist ⁵, this $N=4$ extension turns out to be equivalent to the $N=2$ formulation, as has been demonstrated by the computation of scattering amplitudes [20]. It should be stressed that, being quantum mechanically equivalent to the $N=2$ string after the twisting, the $N=4$ topological formalism offers a larger freedom in formulating a heterotic theory. Since prior to the twisting there are two global $U(1,1)$ groups available, it seems reasonable to preserve *at least one* when adding a left-moving sector. In the next section we shall propose the left-movers and establish explicit space-time supersymmetry. Interestingly enough, we shall find that exactly this type of scenario takes place and the left movers are compatible with *only one* of

⁵It is worth recalling that a topological twist by J implemented in Ref. [20] does not treat all the currents on equal footing. The twisting chooses a $U(1)$ subgroup of the $SU(1,1)$, thus breaking the full Lorentz group $SO(2,2)$ to $U(1,1)$.

the $U(1, 1)$ groups intrinsic to the $N=4$ topological formalism and they explicitly violate the other. To put it in other words, one could proceed with the ordinary $N=2$ string taken to describe right movers. These are invariant under $U(1, 1)$. Then one could add manifestly supersymmetric left movers which prove to be invariant under another $U(1, 1)$. What happens is that each of the $U(1, 1)$ groups leaves only one chiral half invariant at a time and not the full formalism. A remarkable fact, however, is that when acting on the $N=2$ superconformal currents, the left $U(1, 1)$ automatically generates the “small” $N=4$ superconformal algebra. In order to keep a single $U(1, 1)$ for the whole string one is forced to turn to the topological description by Berkovits and Vafa, the global symmetry group of the full formalism being the $U(1, 1)$ coming with the left movers. We thus conclude that, together with the Hamiltonian (17), the currents (18) and (23) completely specify the right-moving sector.

3. The left-moving sector

Since the string coordinates are automatically decomposed into right and left modes, they are already present in the left sector. In the spirit of the GS string, we add space-time spinors which are world-sheet scalars. In spite of the heterotic construction adopted in this paper, we still choose to keep a balance between left- and right-moving degrees of freedom and introduce two canonical pairs,

$$\{\theta^{(+)}(\sigma), p_{(+)}(\sigma')\} = \delta(\sigma - \sigma'), \quad \{\theta^{(-)}(\sigma), p_{(-)}(\sigma')\} = \delta(\sigma - \sigma'), \quad (24)$$

where $(\theta^{(+)}, \theta^{(-)})$ are real and $(p_{\theta^{(+)}} , p_{\theta^{(-)}})$ are imaginary. The indices (\pm) signify weights equal to $\pm\frac{1}{2}$ with respect to the $SO(1, 1)$ subgroup of the full $U(1, 1)$ group and are viewed as would-be spinor indices. In this space one immediately observes the intriguing possibility to have a $c=0$ conformal field theory, matching perfectly the one for the right-movers.

The dynamics of the new fields gets fixed by adding two new terms to the Hamiltonian (17) to arrive at

$$H = \int_0^{2\pi} \left\{ -4\pi p^+ p^- - 4\pi p_Y^+ p_Y^- - \frac{1}{\pi} \partial_1 X^+ \partial_1 X^- - \frac{1}{\pi} \partial_1 Y^+ \partial_1 Y^- + i\varphi^+ \partial_1 \varphi^- + \right. \\ \left. i\varphi^- \partial_1 \varphi^+ + i\chi^+ \partial_1 \chi^- + i\chi^- \partial_1 \chi^+ - p_{(+)} \partial_1 \theta^{(+)} - p_{(-)} \partial_1 \theta^{(-)} \right\}. \quad (25)$$

This yields the free equations of motion

$$\partial_+ \theta^{(\pm)} = 0, \quad \partial_+ p_{\theta^{(\pm)}} = 0. \quad (26)$$

Since the GS-type scalars are single-valued on the world-sheet, they must be taken to be periodic functions on the cylinder.

The set of currents we postulate in the sector is

$$T = -2(p^- - \frac{1}{2\pi} \partial_1 X^-)(p^+ - \frac{1}{2\pi} \partial_1 X^+) - 2(p_Y^- + \frac{1}{2\pi} \partial_1 Y^-)(p_Y^+ + \frac{1}{2\pi} \partial_1 Y^+) - \\ \frac{1}{\pi} p_{(+)} \partial_1 \theta^{(+)} - \frac{1}{\pi} p_{(-)} \partial_1 \theta^{(-)} = 0,$$

$$\begin{aligned}
G &= (p^- - \frac{1}{2\pi}\partial_1 X^-)p_{(-)} = 0, \quad H = (p_Y^- + \frac{1}{2\pi}\partial_1 Y^-)p_{(-)} = 0, \\
J &= p_{(+)}p_{(-)} = 0,
\end{aligned} \tag{27}$$

satisfying

$$\partial_+ T = 0, \quad \partial_+ G = 0, \quad \partial_+ H = 0, \quad \partial_+ J = 0. \tag{28}$$

It is relevant to note that the constraints proposed look very similar to those studied in [14]. They differ, however, in the global symmetry structure. The currents studied in [14] support the full Lorentz group $SO(2, 2)$, while in the present formalism we are forced to stick with a $U(1, 1)$ subgroup.

The only non-vanishing brackets in the corresponding algebra are

$$\begin{aligned}
\{T(\sigma), T(\sigma')\} &= -\frac{2}{\pi}T(\sigma)\partial_1\delta(\sigma - \sigma') - \frac{1}{\pi}\partial_1 T(\sigma)\delta(\sigma - \sigma'), \\
\{G(\sigma), T(\sigma')\} &= -\frac{2}{\pi}G(\sigma)\partial_1\delta(\sigma - \sigma') - \frac{1}{\pi}\partial_1 G(\sigma)\delta(\sigma - \sigma'), \\
\{H(\sigma), T(\sigma')\} &= -\frac{2}{\pi}H(\sigma)\partial_1\delta(\sigma - \sigma') - \frac{1}{\pi}\partial_1 H(\sigma)\delta(\sigma - \sigma'), \\
\{J(\sigma), T(\sigma')\} &= -\frac{2}{\pi}J(\sigma)\partial_1\delta(\sigma - \sigma') - \frac{1}{\pi}\partial_1 J(\sigma)\delta(\sigma - \sigma'),
\end{aligned} \tag{29}$$

implying that all the currents carry conformal spin 2. Two comments are in order. Firstly, the set constructed is invariant under $U(1, 1)$ (see Sect. 4 for the explicit realization). Secondly, making use of arguments like those exploited in [14] one can show that the set above is *functionally dependent* which brings serious problems when one tries to apply BRST techniques. This may be seen by evaluating the functional determinant of the matrix of first derivatives of the constraints with respect to all the variables in the problem: its rank equals two. Unfortunately, choosing an irreducible subset would break the $U(1, 1)$ symmetry. Interestingly, this seems to be analogous to the situation in the conventional $d=10$ Green–Schwarz superstring.⁶

Let us turn to the issue of global supersymmetry. Since, by the very construction, the subspace spanned by $(X^+, \theta^{(+)})$ is nothing other than a $(1, 0)$ –superspace (see Ref. [26] for our definition), it is straightforward to realize the following *on-shell* global supersymmetry,

$$\begin{aligned}
\delta\theta^{(+)} &= \epsilon^{(+)}, \quad \delta X^+ = i\theta^{(+)}\epsilon^{(+)}, \\
\delta p_{(+)} &= -2i(p^- - \frac{1}{2\pi}\partial_1 X^-)\epsilon^{(+)}, \quad \delta p^+ = -\frac{i}{2\pi}\partial_1\theta^{(+)}\epsilon^{(+)}.
\end{aligned} \tag{30}$$

As usual, the corresponding current is defined via the Poisson bracket,

$$\delta A = \{A, \int_0^{2\pi} d\sigma' q_{(+)}(\sigma')\}\epsilon^{(+)}, \tag{31}$$

which yields

$$q_{(+)} = p_{(+)} - 2i\theta^{(+)}(p^- - \frac{1}{2\pi}\partial_1 X^-), \quad \{q_{(+)}(\sigma), q_{(+)}(\sigma')\} = 2iP^-\delta(\sigma - \sigma'), \tag{32}$$

⁶The implication of computing conventional constraints in terms of dual fields for the type-II Green–Schwarz theory has been considered in [24]. For a detailed discussion about the covariant quantization and kappa symmetry, see e.g. [25].

with $P^- \equiv -2(p^- - \frac{1}{2\pi}\partial_1 X^-)$ being the generator of translations in the X^+ -direction.

Due to the equation

$$\partial_+ q_{(+)} = 0, \quad (33)$$

the integral

$$Q_{(+)} = \int_0^{2\pi} d\sigma q_{(+)}(\sigma) \quad (34)$$

gives a conserved charge. The supersymmetry transformations defined above leave the constraint surface (strongly) invariant, as they should.

Finally, we give the classical solutions in the sector (recall that $p_{\theta(+)}$ and $p_{\theta(-)}$ are imaginary):

$$\begin{aligned} \theta^{(\pm)} &= \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} d_n^{(\pm)} e^{-in(\tau-\sigma)}, \quad p_{(\pm)} = \frac{i}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \ell_{n(\pm)} e^{-in(\tau-\sigma)}, \\ \{d_n^{(\pm)}, \ell_{m(\pm)}\} &= -i\delta_{n+m,0}, \quad (d_n^{(\pm)})^* = d_{-n}^{(\pm)}, \quad (\ell_{n(\pm)})^* = \ell_{-n(\pm)}. \end{aligned} \quad (35)$$

To conclude the section, it is worth noting that the structure of the left-moving “spinors” resembles what one usually expects of a ghost system. Each pair $(\theta^{(+)}, p_{(+)})$, $(\theta^{(-)}, p_{(-)})$ makes a contribution of -2 to the conformal anomaly, and altogether they cancel $c=4$ coming from the X^\pm, Y^\pm matter system. Also one suspects them to bring about a degeneracy of the vacuum. We shall turn to this issue later in Sect. 6, where we shall make explicit use of this degeneracy in establishing global supersymmetry of the string spectrum.

4. Global symmetry structure

Having formulated both left and right-movers, we are now in a position to discuss the global symmetry structure of the full theory. For the right-movers it is trivial to transform the common $U(1,1)$ transformations to the real notation. We gather them in the following table.

	X^+	X^-	Y^+	Y^-	φ^+	φ^-	χ^+	χ^-
δ_α	αX^+	$-\alpha X^-$	αY^+	$-\alpha Y^-$	$\alpha \varphi^+$	$-\alpha \varphi^-$	$\alpha \chi^+$	$-\alpha \chi^-$
δ_β	βY^+	βY^-	$-\beta X^+$	$-\beta X^-$	$\beta \chi^+$	$\beta \chi^-$	$-\beta \varphi^+$	$-\beta \varphi^-$
$\delta_{\gamma^{++}}$	$\gamma^{++} Y^-$	0	$-\gamma^{++} X^-$	0	$\gamma^{++} \chi^-$	0	$-\gamma^{++} \varphi^-$	0
$\delta_{\gamma^{--}}$	0	$\gamma^{--} Y^+$	0	$-\gamma^{--} X^+$	0	$\gamma^{--} \chi^+$	0	$-\gamma^{--} \varphi^+$

Table 1. Global $U(1,1)$ transformations acting in the right sector.

These leave invariant the equations of motion and map the $N=2$ superconformal currents into each other. The corresponding conserved charges, obviously, form a $u(1,1)$

algebra. Being symmetries of the right sector, the transformations, however, do not hold in the left sector as they do not leave the currents (27) invariant. However, in the left sector one finds an independent $u(1,1)$ realized in the following way.

	X^+	X^-	Y^+	Y^-	$\theta^{(+)}$	$p_{(+)}$	$\theta^{(-)}$	$p_{(-)}$
δ_ω	ωX^+	$-\omega X^-$	0	0	$\frac{1}{2}\omega\theta^{(+)}$	$-\frac{1}{2}\omega p_{(+)}$	$-\frac{1}{2}\omega\theta^{(-)}$	$\frac{1}{2}\omega p_{(-)}$
δ_ζ	0	0	ζY^+	$-\zeta Y^-$	$\frac{1}{2}\zeta\theta^{(+)}$	$-\frac{1}{2}\zeta p_{(+)}$	$-\frac{1}{2}\zeta\theta^{(-)}$	$\frac{1}{2}\zeta p_{(-)}$
δ_λ	0	λY^-	$-\lambda X^+$	0	0	0	0	0
δ_μ	μY^+	0	0	$-\mu X^-$	0	0	0	0

Table 2. Global $U(1,1)$ transformations acting in the left sector.

We indeed verify that the associated generators,

$$\begin{aligned}
A_1 &= -2p^- X^+ + 2p^+ X^- + \frac{1}{2}p_{(+)}\theta^{(+)} - \frac{1}{2}p_{(-)}\theta^{(-)}, \\
A_2 &= 2p_Y^- Y^+ - 2p_Y^+ Y^- + \frac{1}{2}p_{(+)}\theta^{(+)} - \frac{1}{2}p_{(-)}\theta^{(-)}, \\
A_3 &= -2p^+ Y^- - 2p_Y^- X^+, \quad A_4 = -2p^- Y^+ - 2p_Y^+ X^-,
\end{aligned} \tag{36}$$

lead to the conserved charges

$$\begin{aligned}
L_1 &= -i \int d\sigma (A_1 + A_2), \quad L_2 = -i \int d\sigma (A_1 - A_2), \\
L_3 &= -i \int d\sigma (A_3 + A_4), \quad L_4 = -i \int d\sigma (A_3 - A_4),
\end{aligned} \tag{37}$$

which obey a $u(1,1)$ algebra. Like the transformations discussed above, these symmetries do not hold for the other sector. The crucial observation, however, is that extending the charge L_1 by two new contributions (which do not spoil the algebra!),

$$L_1 \rightarrow L'_1 = -i \int d\sigma (A_1 + A_2 + 2i\varphi^- \varphi^+ + 2i\chi^- \chi^+), \tag{38}$$

one arrives at the transformations which map the currents \tilde{G}, \tilde{H} of the right sector onto \tilde{G}_1, \tilde{H}_1 and vice versa. Then the closure of the algebra automatically produces the remaining currents \tilde{J}_1, \tilde{J}_2 of the $N=4$ topological formalism.

Thus, we see that by adopting the $N=4$ topological description for the right-movers, one can have a unique $U(1,1)$ global group acting on the right- and left-movers simultaneously while keeping space-time supersymmetry in the left sector. It should be stressed that, although there are two $U(1,1)$ groups intrinsic to the $N=4$ formalism, only one of them is compatible with the left-moving sector, while the other is explicitly violated by the latter. The global symmetry group of the whole string is thus a single $U(1,1)$.

In the previous section, the global supersymmetry transformation was specified. Adjoining it to the $U(1,1)$ transformations, the larger algebra closes upon adding two new transformations with the fermionic parameters $\rho^{(+)}$ and $\kappa^{(+)}$,

$$\begin{aligned}\delta_\rho Y^+ &= i\theta^{(+)}\rho^{(+)}, \quad \delta_\rho p_Y^+ = \frac{i}{2\pi}\partial_1\theta^{(+)}\rho^{(+)}, \quad \delta_\rho p_{(+)} = 2i(p_Y^- + \frac{1}{2\pi}\partial_1 Y^-)\rho^{(+)}, \\ \delta_\rho\theta^{(+)} &= 0; \quad \delta_\kappa\theta^{(+)} = \kappa^{(+)}. \end{aligned} \quad (39)$$

It has to be mentioned that these leave invariant the currents in the right sector. The corresponding conserved charges are given by

$$S_{(+)} = 2i \int d\sigma(p_Y^- + \frac{1}{2\pi}\partial_1 Y^-)\theta^{(+)}, \quad \tilde{S}_{(+)} = \int d\sigma p_{(+)}. \quad (40)$$

Taking into account the additional charges associated with the translation invariance $\delta X^\pm = a^\pm$ and $\delta Y^\pm = b^\pm$,

$$B^\pm = -2 \int d\sigma(p^\pm - \frac{1}{2\pi}\partial_1 X^\pm), \quad B_Y^\pm = 2 \int d\sigma(p_Y^\pm + \frac{1}{2\pi}\partial_1 Y^\pm), \quad (41)$$

one can finally write down a complete superalgebra. Most compactly, this is represented by the following table (equal-time brackets involving B^\pm , B_Y^\pm vanish and are omitted here):

$\{, \}$	Q	S	\tilde{S}	L_1	L_2	L_3	L_4
Q	$2iB^-$	iB_Y^-	iB^-	$i\tilde{S}$	$i(Q - \tilde{S})$	$-iS$	$-iS$
S	iB_Y^-	0	iB_Y^-	0	$-iS$	$i(Q - \tilde{S})$	$-i(Q - \tilde{S})$
\tilde{S}	iB^-	iB_Y^-	0	$i\tilde{S}$	0	0	0
L_1	$-i\tilde{S}$	0	$-i\tilde{S}$	0	0	0	0
L_2	$-i(Q - \tilde{S})$	iS	0	0	0	$2iL_4$	$2iL_3$
L_3	iS	$-i(Q - \tilde{S})$	0	0	$-2iL_4$	0	$2iL_2$
L_4	iS	$i(Q - \tilde{S})$	0	0	$-2iL_3$	$-2iL_2$	0
B^+	0	0	0	$-iB^+$	$-iB^+$	$-iB_Y^+$	iB_Y^+
B^-	0	0	0	iB^-	iB^-	$-iB_Y^-$	$-iB_Y^-$
B_Y^+	0	0	0	$-iB_Y^+$	iB_Y^+	iB^+	iB^+
B_Y^-	0	0	0	iB_Y^-	$-iB_Y^-$	iB^-	$-iB^-$

Table 3. The supersymmetry algebra of the full model.

It is straightforward to verify that the Jacobi identities hold for this superalgebra. Note further that the bracket of the $S_{(+)}$ charge with itself is zero. The same is true for $\tilde{S}_{(+)}$. Because the generators are composed of “spinors”, at the quantum level they will be represented by nilpotent operators, failing the usual unitarity argument to show that these symmetries are trivial. In Sect. 6 we will see that $S_{(+)}$, $\tilde{S}_{(+)}$ indeed act on a quantum space, although not playing a significant role.

Before closing this section, it seems instructive to inspect the above symmetry transformations in complex notation. Putting z^a and \bar{z}^a in a single row $Z^A = (z^a, \bar{z}^a)$ with $A=1, 2, 3, 4$, one can conveniently rewrite the (infinitesimal) transformations gathered in Table 1 as follows,

$$\delta Z^A = i\alpha^i L_i^A{}_B Z^B, \quad \bar{L}_i^T \eta = \eta L_i, \quad \eta_{AB} = \text{diag}(-, +, -, +), \quad (42)$$

where the matrices L_i form a basis for the $u(1, 1)$ algebra,

$$\begin{aligned} L_1 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}, & L_2 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \\ L_3 &= \begin{pmatrix} -1 & +1 & 0 & 0 \\ -1 & +1 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ 0 & 0 & +1 & -1 \end{pmatrix}, & L_4 &= \begin{pmatrix} -1 & -1 & 0 & 0 \\ +1 & +1 & 0 & 0 \\ 0 & 0 & +1 & +1 \\ 0 & 0 & -1 & -1 \end{pmatrix}. \end{aligned} \quad (43)$$

Analogously, for the transformations from Table 2 one finds another set of basis elements,

$$\begin{aligned} L_1 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}, & L_2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \\ L_3 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, & L_4 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (44)$$

We see from here that in the former case the matrices are block-diagonal. In other words, the fields z^a and \bar{z}^a do not mix under these transformations. In the latter case, some of the generators are off-diagonal, mixing z^a with \bar{z}^a . This means, in particular, that a combination of the form $\bar{z}^a \eta_{ab} y^b$, which is trivially invariant under the action of the right $U(1, 1)$, does not hold invariant under the action of the left $U(1, 1)$, and a more general object like $\bar{Z}^A \eta_{AB} Y^B$ is to be handled with. As we shall discuss in Sect. 7, this causes certain problems in constructing scattering amplitudes because it prevents one from a naive use of the vertex operator known for the conventional $N=2$ string.

5. Quantized right-movers

As has been discussed in Sect. 2, the right-moving sector of our model relies on the $N=4$ topological extension of (a chiral half of) the $N=2$ string. At the level of string amplitudes a detailed proof of equivalence of the two formalisms was given in [20, 21]. Alternatively, one could proceed directly from the small $N=4$ superconformal algebra to verify that the positive (half-integer) modes of \tilde{G}_1 , \tilde{H}_1 kill all physical states, provided so do the zero modes of \tilde{J}_1 and \tilde{J}_2 (see also the discussion in Ref. [23]). Since even for the smaller $N=2$ superconformal algebra a proper analysis shows the absence of excited states and because the zero modes of \tilde{J}_1 and \tilde{J}_2 annihilate the ground state, one arrives at the same spectrum as for the ordinary $N=2$ string. Below we briefly sketch the main points (conformed to our notation). A more detailed exposition can be found in [6, 20].

Given the matter system $X^\pm, Y^\pm, \varphi^\pm, \chi^\pm$ one finds $c=6$ for the conformal anomaly. Introducing the superconformal ghosts $(c, b), (\gamma, \beta), (\gamma_1, \beta_1), (c_1, b_1)$ associated with the $N=2$ superconformal currents $\tilde{T}, \tilde{G}, \tilde{H}, \tilde{J}$, the ghost contribution to the anomaly adds up to $-26 + 11 + 11 - 2 = -6$. Thus, this sector is critical in $2+2$ dimensions. Evaluating the normal-ordering constants one finds a vanishing critical intercept so that the ground state is a massless scalar. The BRST analysis shows that the latter is in fact the only physical state in the spectrum [27, 6]. It is created from the (zero-momentum) NS vacuum by application of a vertex operator whose form depends on the superconformal ghost picture [28] chosen. Abbreviating $K^\pm = k^\pm + i\kappa^\pm$ and suppressing the ghost structure, one has in the (-1) picture

$$V_{-1}(K, z) = : e^{i(K\bar{z} + \bar{K}z)} :=: e^{-2i(k^+X^- + k^-X^+ + \kappa^+Y^- + \kappa^-Y^+)} :, \quad (45)$$

while the 0-picture vertex operator reads (conformed to our notation)

$$V_0(K, z) = : 2(iK\partial\bar{z} - i\bar{K}\partial z - 2\pi\bar{K}\psi K\bar{\psi}) e^{i(K\bar{z} + \bar{K}z)} :. \quad (46)$$

Further analysis shows that only the three-point function does not vanish at tree level. In the zero-instanton sector one finds⁷

$$\begin{aligned} A_{right}^{tree}(1, 2, 3) &= \langle V_{-1}(K_1, \infty) V_0(K_2, 1) V_{-1}(K_3, 0) \rangle \\ &= K_2 \bar{K}_3 - \bar{K}_2 K_3 \\ &= 2i(k_2^+ \kappa_3^- + k_2^- \kappa_3^+) - 2i(k_3^+ \kappa_2^- + k_3^- \kappa_2^+). \end{aligned} \quad (47)$$

Due to the relation between open- and closed-string amplitudes [29], one might expect that Eq. (47) is the only contribution to the full S-matrix coming from the right sector. However, the story is not so simple because, upon a more careful inspection, one observes that Eq. (46) is not invariant under the global $U(1, 1)$ group installed by the left-movers. In other words, disregarding the full set of $N=4$ superconformal currents and constructing the interactions by taking into account only the $N=2$ subset breaks the manifest $U(1, 1)$ covariance of the full formalism. We will return to this issue in Sect. 7.

⁷The three-point functions in the other instanton sectors are proportional to this one.

6. Quantized left-movers

As we have seen in Sect. 3, the constraints intrinsic to the left-movers are (infinitely) reducible. This entails a serious complication for the BRST procedure (for a similar point see e.g. [14]). For this reason we employ the conventional operator method for the covariant quantization.

Because bosonic fields are common for both right and left sectors, it suffices to discuss the “spinors”. The zero-mode algebra⁸

$$\{d_0, \ell_0\} = 1, \quad d_0^2 = 0 = \ell_0^2 \quad (48)$$

enforces a two-fold degeneracy of the vacuum. The fields $\theta^{(-)}$ and $p_{(-)}$ are handled in precisely the same manner, so the full vacuum is four-fold degenerate. In what follows, we discuss in detail a representation for $\theta^{(+)}, p_{(+)}$ only and for brevity omit the $(+)$ index:

$$\begin{aligned} \ell_0 | \downarrow \rangle &= 0, \quad \ell_0 | \uparrow \rangle = | \downarrow \rangle, \quad d_0 | \downarrow \rangle = | \uparrow \rangle, \quad d_0 | \uparrow \rangle = 0, \\ \ell_n | \downarrow \rangle &= \ell_n | \uparrow \rangle = d_n | \downarrow \rangle = d_n | \uparrow \rangle = 0 \quad \text{for } n \geq 1. \end{aligned} \quad (49)$$

Since we are interested in a unitary representation and $d_0^\dagger = d_0, \ell_0^\dagger = \ell_0$, the scalar products are

$$\langle \downarrow | \downarrow \rangle = 0, \quad \langle \uparrow | \uparrow \rangle = 0, \quad \langle \downarrow | \uparrow \rangle = \langle \downarrow | \theta | \downarrow \rangle = 1. \quad (50)$$

These relations, as well as the value of the conformal spin carried by the “spinors”, identify (θ, p) as a fermionic $c=-2$ system. The latter arises as R symmetry ghosts in the conventional $N=2$ string [2] and as auxiliary fermions in the bosonization of the superconformal ghosts of the $N=1$ NSR string [28].

As for the right-movers, the normal-ordering constants add to zero. The quantum version of the algebra (29),

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m}, \\ [L_n, G_m] &= (n-m)G_{n+m}, \\ [L_n, H_m] &= (n-m)H_{n+m}, \\ [L_n, J_m] &= (n-m)J_{n+m}, \end{aligned} \quad (51)$$

implies that physical states in the complete Hilbert space are defined by

$$L_n |\text{phys}\rangle = G_n |\text{phys}\rangle = H_n |\text{phys}\rangle = J_n |\text{phys}\rangle = 0 \quad \text{for } n \geq 0. \quad (52)$$

Because, by the very construction, there are no excited physical states in the right sector, it suffices to concentrate on the left-sector ground state. As we have seen above this is four-fold degenerate,

$$\Phi(k, \kappa) | \downarrow \rangle \times | \downarrow \rangle, \quad \Psi_{(+)}(k, \kappa) | \uparrow \rangle \times | \downarrow \rangle, \quad \Upsilon_{(-)}(k, \kappa) | \downarrow \rangle \times | \uparrow \rangle, \quad \Sigma(k, \kappa) | \uparrow \rangle \times | \uparrow \rangle. \quad (53)$$

⁸From now on, $[,]$ and $\{, \}$ denote commutators and anticommutators, respectively.

Recall that the operators $d_0^{(\pm)}$, which raise $|\downarrow\rangle \rightarrow |\uparrow\rangle$, carry weights of $\pm\frac{1}{2}$ with respect to the $SO(1,1)$ subgroup of the full $U(1,1)$ group and, hence, assign those to the excited states $|\uparrow\rangle^{(\pm)}$. For brevity we omit indices carried by the states throughout the text.

On the ground states (53), the conditions (52) reduce to their zero-mode part, with

$$L_0 \rightarrow -2(p^- p^+ + p_y^- p_y^+), \quad G_0 \rightarrow p^- \ell_{0(-)}, \quad H_0 \rightarrow p_y^- \ell_{0(-)}, \quad J_0 \rightarrow \ell_{0(+)} \ell_{0(-)}. \quad (54)$$

As usual, L_0 forces the states to lie on the mass shell

$$k^+ k^- + \kappa^+ \kappa^- = 0. \quad (55)$$

The other three operators are nilpotent. The J_0 condition eliminates Σ . The remaining constraints G_0 and H_0 imply

$$k^- \Upsilon_{(-)} = \kappa^- \Upsilon_{(-)} = 0 \quad (56)$$

which, for generic kinematics (normalizability of states), renders $\Upsilon_{(-)}$ unphysical as well.

Hence, we are left with the pair $(\Phi, \Psi_{(+)})$ for the time being. A relevant point to check is whether these two remaining massless states are indeed connected by supersymmetry. Taking the Laurent expansion of the supersymmetry current $q_{(+)}$ (32) in the form

$$q_{(+)}(z) = \frac{1}{\sqrt{2\pi^2}} \sum_{n \in \mathbb{Z}} \frac{q_n}{z^{n+1}} \quad (57)$$

and evaluating the operator products with the superconformal currents (27), one easily finds the commutation relations

$$\begin{aligned} [L_n, q_m] &= -m q_{n+m}, & \{G_n, q_m\} &= 0, \\ \{H_n, q_m\} &= 0, & [J_n, q_m] &= -\frac{i\sqrt{2}}{\pi} G_{n+m}. \end{aligned} \quad (58)$$

One sees from here that only the zero mode of $q_{(+)}$, namely

$$q_{0(+)} = i\sqrt{\pi} \ell_{0(+)} - \frac{i}{\sqrt{\pi}} \sum_{n \in \mathbb{Z}} d_n^{(+)} a_{-n}^-, \quad (59)$$

(weakly) commutes with the super Virasoro generators and, hence, maps physical states to physical states. For the ground states at hand this yields

$$\delta_{\text{susy}} \Phi = i\sqrt{\pi} \epsilon^{(+)} \Psi_{(+)}, \quad \delta_{\text{susy}} \Psi_{(+)} = -\frac{i}{\sqrt{\pi}} \epsilon^{(+)} p^- \Phi, \quad (60)$$

and the corresponding fields do fall into a supermultiplet.

Let us finally discuss the way the $U(1,1)$ symmetry acts on the physical states or, equivalently, on the vertex operators. Since the analysis reduces to the zero-mode sector, one can readily deduce from (37) a quantum representation for the $u(1,1)$ algebra (the ordering of the operators involved is fixed as given below)

$$\begin{aligned} L_1 &= 2x^+ p^- - 2x^- p^+ + 2y^+ p_y^- - 2y^- p_y^+ + id_0^{(+)} \ell_{0(+)} - id_0^{(-)} \ell_{0(-)}, \\ L_2 &= 2x^+ p^- - 2x^- p^+ - 2y^+ p_y^- + 2y^- p_y^+, \\ L_3 &= 2y^- p^+ - 2x^+ p_y^- + 2y^+ p^- - 2x^- p_y^+, \\ L_4 &= 2y^- p^+ - 2x^+ p_y^- - 2y^+ p^- + 2x^- p_y^+, \end{aligned} \quad (61)$$

were $\{d_0, \ell_0\} = 1$, $[x^\pm, p^\mp] = -\frac{i}{2}$, and $[y^\pm, p_y^\mp] = -\frac{i}{2}$. Notice that the operators generating the algebra are hermitian. One sees from the relations above that the generators L_2, L_3, L_4 forming the $su(1,1)$ subalgebra (see Table 3 for the commutation relations) do not involve any spin part. This means that the physical states are singlets with respect to the $SU(1,1)$ subgroup of the full $U(1,1)$. The generator L_1 of the abelian subalgebra $u(1)$, however, contains a nontrivial “spin” part $d_0^{(+)}\ell_{0(+)}$ which measures the helicity of our massless states Φ and $\Psi_{(+)}$ as 0 and $1/2$, respectively, and thus distinguishes between them. In contrast to the ordinary $N=2$ string, where states of different helicity are physically equivalent due to the action of picture-raising and spectral flow, Φ and $\Psi_{(+)}$ represent inequivalent states because of their left-moving structure.

7. Discussion

To summarize, in the present paper we examined a novel possibility for including space-time fermions into the $N=2$ string. In contrast to previous approaches we chose to maintain the $U(1,1)$ covariance intrinsic to the original $N=2$ string and supersymmetrized only one direction of the configuration space. Supersymmetry has been realized explicitly on the left-movers. Interestingly enough, application of the $U(1,1)$ transformations to the standard set of $N=2$ superconformal currents, which was our starting point in the right-moving sector, automatically generates the $N=4$ currents of the topological reformulation by Berkovits and Vafa. It seems that the space-time supersymmetry distinguishes the latter from the former. Quantization has been accomplished, yielding two physical states in the spectrum, one boson and one fermion, which form a multiplet of the global $(1,0)$ supersymmetry.

We turn to discuss some points which have not been covered in the paper and may constitute further developments. Firstly, since from the outset Lorentz covariance has been sacrificed, the question of spin carried by the quantum states in the spectrum can be answered only after an explicit evaluation of tree-level scattering amplitudes. However, this immediately reveals a difficulty because the (right-sector) vertex operator (46) does not respect the $U(1,1)$ covariance of the full formalism which is kept by the left-movers (see Sect. 4). Secondly, the computation of scattering amplitudes requires a determination of the (θ, p_θ) zero-mode measure in the vertex operator correlation functions. As is familiar from ghost systems, such zero-mode insertions can be subsumed in a modification of the scalar product (50) plus a change of the conjugation rules. Also, the full vertex operators V_0 are yet to be constructed. These questions require a further investigation and we hope to report on them elsewhere. Thirdly, throughout the paper we worked in the superconformal gauge. A Lagrangian formulation is missing and suggests another interesting problem. Yet, from the structure of the constraints one may already suspect that a Lagrangian formulation is likely to possess both kappa symmetry (acting on the left-moving fields) as well as an $N=2$ world-sheet local supersymmetry (acting on the right-movers). This could provide an interesting combination of Green-Schwarz and Neveu-Schwarz-Ramond formalisms.

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